Exam Lie Groups in Physics

Date	November 8, 2021
Room	Exam Hall 1
Time	08:30 - 11:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	7	2a)	9	3a)	6	4a)	9
1b)	6	2b)	9	3b)	6	4b)	6
1c)	7	2c)	6	3c)	6	4c)	7
				3d)	6		

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Problem 1

(a) Provide the definition of a Lie algebra.

(b) Write down the properties of a homomorphism between two Lie algebras.

(c) Show that the Lie algebra of an invariant subgroup H of a group G forms an invariant subalgebra L_H of the Lie algebra L_G of G.

Problem 2

Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series:

$$\square\square \otimes \square\square \otimes \square\square$$

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

(c) Relate the above decomposition for $su(2)$ to the addition of 3 angular momentum states in Quantum Mechanics. Write this explicitly in terms of eigenvalues of the 3 angular momentum states to be added and of the total angular momentum.

Problem 3
Consider the group $O(2)$ of real orthogonal 2×2 matrices.

(a) Write down all elements of $O(2)$ in its defining representation.

(b) Use Schur's lemma to show that the defining representation of $O(2)$ is irreducible.

(c) Explain why applying Schur's lemma to the generators of $O(2)$ transformations is insufficient to conclude whether or not a representation is irreducible.

(d) Write down a 3-dimensional representation of $O(2)$.

Problem 4
Consider the group $SL(2, \mathbb{C})$ of 2×2 complex matrices with determinant equal to 1 and its relation to the Lorentz group.

(a) Show that

$$H(\chi, \hat{n}) \equiv \exp(-\chi \hat{n} \cdot \vec{\sigma}/2) = \mathbf{1} \cosh(\chi/2) - \hat{n} \cdot \vec{\sigma} \sinh(\chi/2),$$

for any unit vector $\hat{n}$ and real parameter χ. Recall that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy $\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i\epsilon_{ijk}\sigma_k$ $(i = 1, 2, 3)$, and that $\cosh \chi = (e^{\chi} + e^{-\chi})/2$ and $\sinh \chi = (e^{\chi} - e^{-\chi})/2$.

(b) Show that all $H(\chi, \hat{n})$ are elements of $SL(2, \mathbb{C})$.

Next consider the homomorphism from $SL(2, \mathbb{C})$ to the Lorentz group $O(3, 1)$: $A \mapsto L^{\mu}{}_{\nu} = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^{\mu} A \sigma_{\nu} A^{\dagger})$, where $\sigma^{\mu} = (1, \vec{\sigma})$ and $\tilde{\sigma}^{\mu} = (1, -\vec{\sigma})$, such that $\mathrm{Tr}(\sigma^{\mu}\tilde{\sigma}^{\nu}) = 2g^{\mu\nu}$.

(c) Explain why this homomorphism is neither 1-1 (injective) nor onto (surjective) when viewed as a mapping to the full Lorentz group $O(3, 1)$.